Physics 103: Spherical Harmonics Lab Report

Group Members: Evan Benauelos, Remiel Shirazi

Saturday 18th July, 2026

Abstract

We investigated standing waves on a spherical surface and showed how spherical harmonics arise as angular solutions of the wave equation. By reducing the 3D wave equation in spherical coordinates to a 2D surface problem, we derived spherical harmonics $Y_\ell^m(\theta, \varphi)$ as eigenfunctions of the Laplace–Beltrami operator. Representative modes such as monopole, dipole, and quadrupole solutions were analyzed, and plots were created to visualize their nodal patterns. The results demonstrated how multipole language connects mathematical solutions to physical interpretations. Applications in quantum mechanics, geophysics, and computer graphics highlight the significance of spherical harmonics. This lab strengthened our understanding of mathematical methods for solving wave equations and emphasized the broad utility of spherical harmonics across physics and engineering.

1 Introduction

Standing waves on rectangular domains are sinusoidal; on a sphere, the angular modes are spherical harmonics. We studied these modes by deriving them from the wave equation and visualizing low-order solutions. Our objective was to understand how geometry affects standing-wave solutions and to connect the mathematics to physical multipoles. We also point out how the spherical case differs from the familiar sine/cosine modes on rectangles in both nodal structure and symmetry.

2 Theory

2.1 Wave Equation in Three Dimensions

The starting point is the three-dimensional wave equation, which describes how a disturbance $u(r, \theta, \phi, t)$ evolves in time and space:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{1}$$

Here u is the wave function, which may represent displacement of material or pressure; c is the wave speed. The four variables are:

- r = radial distance,
- θ = polar angle (north–south, like latitude),
- ϕ = azimuthal angle (east–west, like longitude),
- t = time.

2.2 Laplacian in Spherical Coordinates

To apply the wave equation to a sphere, we expand the Laplacian in spherical coordinates. It has three distinct contributions:

$$\nabla^2 u = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)}_{\text{radial part}} + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}_{\text{polar part}} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}}_{\text{azimuthal part}}. \tag{2}$$

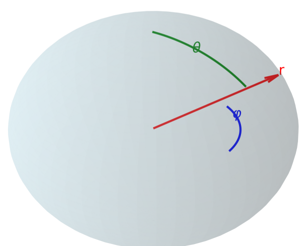

Figure 1: Spherical coordinates: r (radius), θ (polar angle like latitude), ϕ (azimuthal angle like longitude).

2.3 3D vs 2D: Volume vs Area

Before simplifying, it is useful to compare how waves behave in 3D versus 2D:

- In **3D**, body waves spread through volume. As r increases, energy is distributed over spherical shells whose area grows like $4\pi r^2$. Intensity decreases as $1/r^2$.
- In **2D**, surface waves are trapped on the crust. The radius R is fixed, so energy redistributes only across the constant surface area $4\pi R^2$, like ripples on a pond.

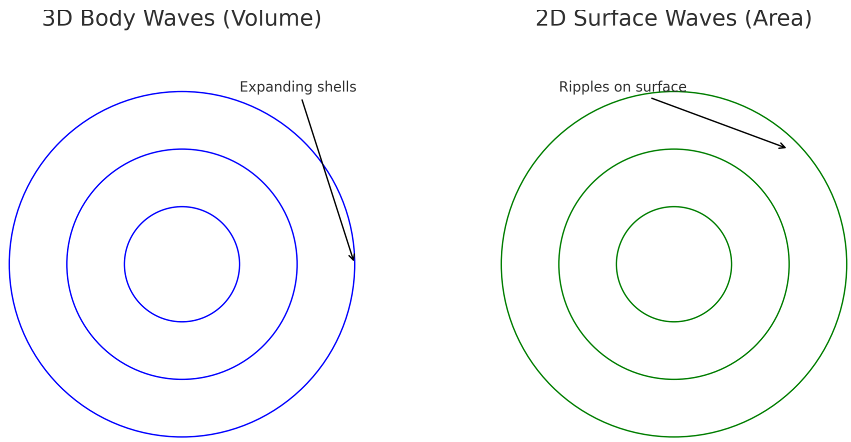

Figure 2: Left: 3D body waves spreading through volume. Right: 2D surface waves spreading across an area of fixed radius.

2.4 Fixing the Radius: Explicit Removal of the Radial Term

We now restrict the physics to the surface of the sphere at $r = R$. Assuming the wave has no radial variation,

$$\frac{\partial u}{\partial r} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}\Big(r^2 \frac{\partial u}{\partial r}\Big) = 0.$$

Therefore the radial part vanishes. The surface Laplacian becomes

$$\nabla^2 u\Big|_{r=R} = \frac{1}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}\right], \tag{3}$$

where $U(\theta, \phi, t) = u(R, \theta, \phi, t)$. This operator is the Laplace–Beltrami operator on the unit sphere.

2.5 Finding Spherical Harmonic Functions

Spherical harmonics are the natural angular "building blocks" for waves on a sphere, much like sine and cosine functions are for rectangular domains. They describe angular standing-wave patterns and are indexed by integers (ℓ, m). We now derive them.

Separation of Variables

Assume separability:

$$U(\theta, \phi) = g(\theta)\, h(\phi). \tag{4}$$

Substitution into the surface Laplacian equation gives

$$\frac{1}{g(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \frac{1}{h(\phi)\sin^2\theta}\frac{d^2h}{d\phi^2} = -\lambda, \tag{5}$$

where λ is the separation constant.

Azimuthal Part

From the ϕ equation:

$$\frac{1}{h(\phi)}\frac{d^2h}{d\phi^2} = -m^2, \tag{6}$$

with solutions

$$h(\phi) = Ae^{im\phi} + Be^{-im\phi}, \tag{7}$$

periodic only if $m \in \mathbb{Z}$.

Polar Part

The θ equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] g(\theta) = 0. \tag{8}$$

Let $x = \cos\theta$, $y(x) = g(\theta)$. Then

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] y = 0, \tag{9}$$

the associated Legendre differential equation.

General Form

Thus

$$Y_\ell^m(\theta,\phi) = N_{\ell m}\, P_\ell^{|m|}(\cos\theta)\, e^{im\phi}. \tag{10}$$

Normalization

Normalization requires

$$\int_0^{2\pi}\int_0^{\pi} |Y_\ell^m|^2 \sin\theta\, d\theta\, d\phi = 1. \tag{11}$$

The ϕ integral gives 2π, while the θ integral uses orthogonality of P_ℓ^m, leading to factorial ratios:

$$\int_0^{\pi} \left[P_\ell^m(\cos\theta)\right]^2 \sin\theta\, d\theta = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}. \tag{12}$$

Thus

$$N_{\ell m} = (-1)^m\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}. \tag{13}$$

Final Expression

$$Y_\ell^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, P_\ell^{|m|}(\cos\theta)\, e^{im\phi}. \tag{14}$$

2.6 Quick Contrast: Rectangular vs Spherical

On a rectangle, separation gives $\sin(k_x x)\sin(k_y y)$. Nodes are straight. On a sphere, Y_ℓ^m has circular nodes: $|m|$ longitudinal circles and $\ell - |m|$ latitudinal circles. Modes are labeled by (ℓ, m), with rotational symmetry mixing m values.

2.7 Multipole Language

Multipole order is tied to ℓ:

- $\ell = 0$ monopole: no nodes.
- $\ell = 1$ dipole: one great-circle node, two lobes.
- $\ell = 2$ quadrupole: two nodes, four alternating lobes.

3 Results and Discussion

Low-order modes were analyzed: Y_0^0, Y_1^m, Y_2^m. Plots show nodal patterns with red for positive, blue for negative.

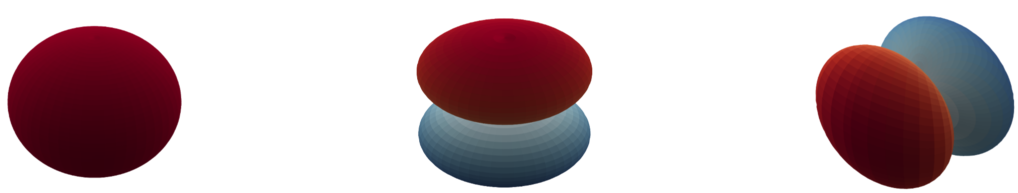

Figure 3: Low-order spherical harmonics: Y_0^0, Y_1^0, and Y_1^1.

4 Applications

Spherical harmonics appear in:

- **Quantum mechanics**: hydrogen atom orbitals.
- **Geophysics**: gravity and geomagnetic field expansions.
- **Computer graphics**: spherical harmonic lighting for fast rendering.

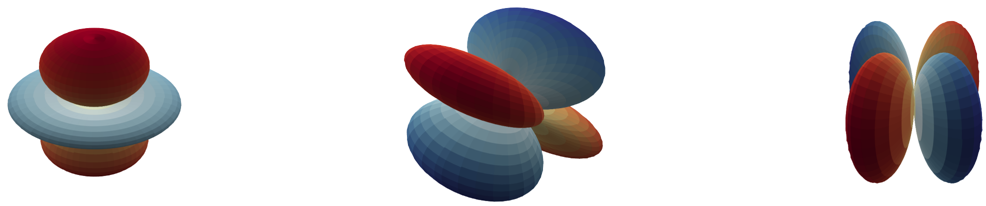

Figure 4: Quadrupole family spherical harmonics: Y_2^0, Y_2^1, and Y_2^2.

5 Conclusion

We derived spherical harmonics by reducing the 3D wave equation to the surface and separating variables. Visualizing low-order modes clarified multipole interpretation. The lab reinforced the role of geometry in shaping wave solutions and the wide use of spherical harmonics in science and engineering.

References

1. OpenStax University Physics, Vol. 1 and Vol. 3. `https://openstax.org/subjects/science`.

2. Haber, H. E. *The Spherical Harmonics.* UCSC. `https://scipp.ucsc.edu/~haber/ph116C/SphericalHarmonics_12.pdf`.

3. StackExchange. "Normalization in spherical harmonics." `https://math.stackexchange.com/questions/1843439/`.